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Modulational Instability of Ion-Acoustic Waves

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The modulational instability of ion-acoustic waves in a collisionless plasma is studied taking into account the effect of ion temperature. It is found that the critical wavenumber is strongly dependent upon the ion temperature. In the limiting case of vanishing ion temperature, we recover the result that the modulational instability sets in for $k > k_c$, where the critical wavenumber is $k_c = 1.47$.

1. INTRODUCTION

The nonlinear wave propagation in a weakly dispersive media has received considerable attention during the last decade. The inclusion of nonlinear terms results in amplitude modulation. In various problems of interest, it has been shown that the long-time slow modulation of wave amplitude is governed by the nonlinear Schrödinger equation (NLSE).

The nonlinear analysis for ion-acoustic waves in a dispersive and weakly nonlinear plasma has been carried out by Ichikawa *et al.* (1972), Schmizu and Ichikawa (1972), Kakutani and Sugimoto (1974), and Mishra *et al.* (1989). Experimental observations of modulational instability were obtained by Watanabe (1977). The modulational instability of ion-acoustic dispersive waves in a collisionless plasma has been studied by Buti (1976) by employing Krylov–Bogoliubov–Mitropolsky (KBM) methods; Buti demonstrated that the critical wavenumber k_c is sensitive to the collisional effect.

Using the KBM method, Durrani *et al.* (1979) studied the effect of ion temperature on the modulational instability of ion-acoustic waves in a collisionless plasma. They showed that the modulational instability sets in much earlier with the inclusion of the ion temperature. In this note, we

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investigate the problem in a systematic manner by employing the method of multiple scales. In Section 2 we derive the dispersion relation for one-dimensional linear waves in a collisionless plasma. The necessary condition for instability is obtained in Section 3.

2. WAVE EQUATIONS

The basic equations governing the propagation of nonlinear ion-acoustic waves for a collisionless plasma consisting of ions and isothermal electrons are (we use the notation $f_{,\alpha} = \partial f/\partial\alpha$)

$$\eta_{,t} + \eta u_{,x} + u\eta_{,x} = 0 \tag{1}$$

$$u_{,t} + uu_{,x} + \frac{\sigma}{n} p_{,x} = E \tag{2}$$

$$p_{,t} + up_{,x} + 3pu_{,x} = 0 \tag{3}$$

$$E_{,x} - n + n_e = 0 \tag{4}$$

$$(n_e)_{,x} + En_e = 0 \tag{5}$$

where n, n_e, u, E, and p denote, respectively, the ion density, the electron density, the ion fluid velocity, the electric field, and the ion pressure. All these physical quantities are normalized, respectively, with undisturbed density, the characteristic sound speed, the characteristic electric field, and the characteristic ion pressure. The parameter σ is the ratio of the ion temperature to the electron temperature.

In order to describe the nonlinear interactions of small but finite amplitude waves, we apply the method of multiple scales by introducing the variables $x_n = \epsilon^n x$, $t_n = \epsilon^n t$, $n = 0, 1, 2$. The small parameter ϵ characterizes the steepness ratio of the wave. We now expand perturbed surfaces $\eta(x, t)$ and all the physical quantities as asymptotic series in ϵ. Then, on substituting the asymptotic series in the basic equations and equating coefficients of like powers in ϵ, we obtain the linear as well as successive higher order equations, each of which can be solved with the knowledge of the solutions of the previous orders.

The linear progressive wave solutions obtained from the first-order equations lead to the dispersion relation

$$D(\omega, k) = -\frac{\omega^2}{k^2} + 3\sigma + \frac{1}{1 + k^2} = 0 \tag{6}$$

Since $\omega^2 > 0$, the first-order solutions represent a uniformly traveling wave-train. We then proceed to the higher order equations, since our aim is to study amplitude modulation of the progressive waves.

3. NONLINEAR SCHRÖDINGER EQUATION

The second-order solutions are uniformly valid, subject to the solvability condition

$$\frac{\partial A}{\partial t_1} + V_g \frac{\partial A}{\partial x_1} = 0 \tag{7}$$

where $V_g = d\omega/dk$ is the group velocity of the wave train, and implies that to the lowest order in ϵ, the amplitude A of the wave packet remains constant in a frame of reference moving with the group velocity of the waves.

In the third order, we obtain the equation governing the complex amplitude modulation. The condition for the surface deflection η_3 to be nonsecular is obtained as

$$i\left(\frac{\partial A}{\partial t_2} + V_g \frac{\partial A}{\partial x_2}\right) + P \frac{\partial^2 A}{\partial x_1^2} = QA^2\bar{A} + RA \tag{8}$$

where

$$P = -\frac{3k^4}{2\omega^3 Q_5^4}[1 + \sigma Q_5(4 - Q_5)] \tag{9}$$

$$Q = (4\omega Q_5)^{-1}[4Q_1^2 Q_2 + (Q_1 + 2Q_5)(Q_1 - Q_5 + Q_6 - 1) - 2Q_3 Q_6 - 3k^2 Q_5] \tag{10}$$

$$R = k^2(2\omega)^{-1}[2Q_1Q_2(c_2 + c_1V_g + Q_4) + 3Q_4 - C_1k^2] \tag{11}$$

$$Q_1 = 1 + Q_5[1 + 12\sigma Q_5] \tag{12}$$

$$Q_2 = [2Q_5(V_g^2 - 3\sigma - 1)]^{-1} \tag{13}$$

$$Q_3 = 1 + \frac{k^2}{Q_5} \tag{14}$$

$$Q_4 = \frac{\sigma(C_3 - 3C_1)}{V_g} \tag{15}$$

$$Q_5 = 1 + k^2 \tag{16}$$

$$Q_6 = (3k^2)^{-1}[3Q_5^2(1 + 4\sigma Q_5) - 1] \tag{17}$$

where the C_i are arbitrary constants. We now introduce the transformations

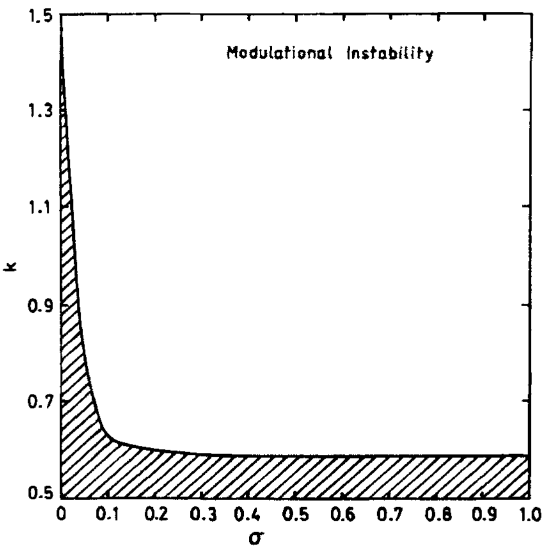

Fig. 1. Stable region (shaded) in the σ–k plane.

$$\xi = \epsilon^{-1}(x_2 - V_g t_2) = x_1 - V_g t_1 = \epsilon(x - V_g t) \tag{18}$$

$$\tau = t_2 = \epsilon t_1 = \epsilon^2 t \tag{19}$$

and

$$A = A \exp\left(-i \int^{\tau} R(\tau')\, d\tau'\right) \tag{20}$$

Then, equation (18) yields

$$i \frac{\partial A}{\partial \tau} + P \frac{\partial^2 A}{\partial \xi^2} = QA^2\bar{A} \tag{21}$$

This equation is the well-known NLSE and its solutions are unstable against modulation if $PQ < 0$. We observe that the coefficients of the dispersive terms P and the nonlinear interaction term Q both depend on σ. Thus the unstable region will be modified due to the effect of ion temperature. We have calculated the critical wavenumber k at which the modulational instability sets in for a given σ. It is interesting to note that the effect of ion temperature on the modulational instability of the ion-acoustic wave is to extend the region of stable wavenumber in the direction of small wavenumbers (see Fig. 1).

REFERENCES

Buti, B. (1976). *IEEE Transactions on Plasma Science,* **4**, 292.

Durrani, I. R., Murtaza, G., and Rahman, H. U. (1979). *Physics of Fluids,* **22**, 791.

Ichikawa, Y. H., Imamira, T., and Taniuti, T. (1972). *Journal of the Physical Society of Japan,* **33**, 189.

Kakutani, T., and Sugimoto, N. (1974). *Physics of Fluids,* **17**, 1617.

Mishra, M. K., Chhabra, R. S., and Sharma, S. R. (1989). *Journal of Plasma Physics,* **42**, 379.

Schmizu, K., and Ichikawa, Y. H. (1972). *Journal of the Physical Society of Japan,* **33**, 789.

Watanabe, S. (1977). *Journal of Plasma Physics,* **17**, 487.